九十七學年第二學期 PHYS2320 電磁學 期末考試題(共二頁)

[Griffiths, Chaps. 9, 10, and 12] 2009/06/09, 10:10am–12:00am, 教師：張存續

記得寫上**學號，班別及姓名**。請依題號順序**每頁答一題**。

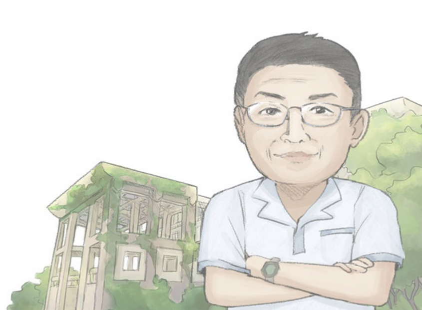
1. (10%, 10%)
   (a) **Einstein's velocity addition rule.** The transformations between two inertial systems S and $\bar{S}$ are $\bar{x}=\gamma(x-vt)$ and $\bar{t}=\gamma(t-vx/c^2)$. v is the relative speed of the system $\bar{S}$ with respect to the system S. Find the relation between the two velocities ($\bar{u}=d\bar{x}/d\bar{t}$ and $u=dx/dt$).
   (b) Two lumps of clay, each of (rest) mass m, collide head-on each at $4/5c$. They stick together. Find the mass (M) of the composite lump? Why the total mass was not conserved?

2. (10%, 10%)
   (a) Show that $(\mathbf{E}\cdot\mathbf{B})$ is relativistically invariant.
   (b) Suppose that in one inertia system $\mathbf{B}=0$ but $\mathbf{E}\neq 0$ (at some point P). Is it possible to find another system in which the electric field is zero at P.

$$\bar{E}_x=E_x,\quad \bar{E}_y=\gamma(E_y-vB_z),\quad \bar{E}_z=\gamma(E_z+vB_y)$$

$$\bar{B}_x=B_x,\quad \bar{B}_y=\gamma(B_y+\frac{v}{c^2}E_z),\quad \bar{B}_z=\gamma(B_z-\frac{v}{c^2}E_y)$$

3. (8%, 12%)
   (a) Write down the Retarded scalar and vector potentials (4%)
   (b) Show that the retarded scalar potential satisfies the inhomogeneous wave equation.



4. (10%, 10%) Consider a coaxial cable, of length d, consisting of an inner conductor (radius a) and an outer conductor (radius b).

(a) If the electric field is $\mathbf{E}(s,\phi,z,t)=\dfrac{A\cos(kz-\omega t)}{s}\hat{\mathbf{s}}$, find the magnetic field and the Poynting vector (i.e. the energy flux).

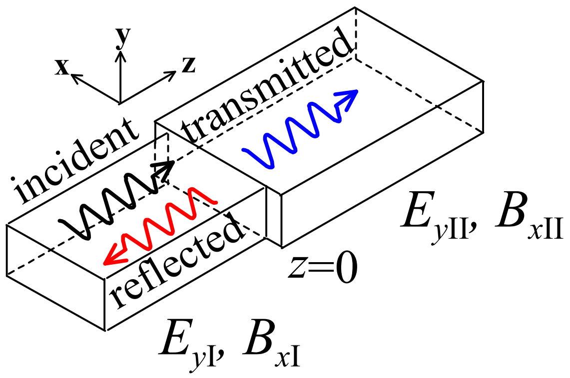
(b) Consider the resonant cavity produced by closing the two ends of the coaxial waveguide. Find the lowest resonant frequency for TEM mode and calculate the Poynting vector.

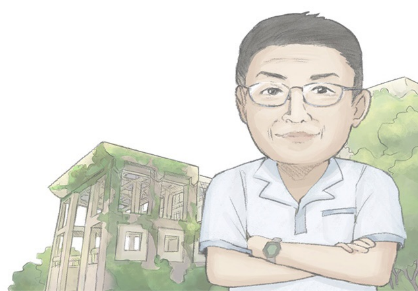
5. (10%, 10%) Consider a TE_{10} wave propagates along the z-axis. The incident wave hits an E-plane discontinuity at z=0 as shown in the figure below.

(a) Write down the electric and magnetic fields in region I and II, respectively.

(b) Calculate the reflection coefficient ($T=$ transmitted power/incident power)

[Hint: Incident wave $\tilde{\mathbf{E}}_I(z,t)=\tilde{E}_{0I}e^{i(k_1z-\omega t)}\hat{\mathbf{y}}$, $\tilde{\mathbf{B}}_I(z,t)=-\dfrac{1}{v_1}\tilde{E}_{0I}e^{i(k_1z-\omega t)}\hat{\mathbf{x}}+(\text{can be omitted})\hat{\mathbf{z}}$].





1.

(a) $\bar{x} = \gamma(x - vt) \Rightarrow d\bar{x} = \gamma(dx - vdt)$,

$\bar{t} = \gamma(t - vx/c^2) \Rightarrow d\bar{t} = \gamma(dt - vdx/c^2)$

$$\frac{d\bar{x}}{d\bar{t}} = \bar{u} = \frac{\gamma(dx - vdt)}{\gamma(dt - vdx/c^2)} = \frac{(u-v)}{(1 - vu/c^2)} \text{ where } \frac{dx}{dt} = u \rightarrow u = \frac{\bar{u} + v}{1 + \bar{u}v/c^2}$$

(b) Example 12.7

before: $E \equiv \dfrac{2mc^2}{\sqrt{1 - u^2/c^2}}$ relativistic energy, $u = \dfrac{4}{5}c$

after: $E \equiv Mc^2 = \dfrac{10mc^2}{3}$ $\therefore\ M = \dfrac{10m}{3} > 2m$

Kinetic energy is converted into rest energy, so the total mass increases.

2. HW 12.46

(a)

$$\bar{E}_x = E_x,\quad \bar{E}_y = \gamma(E_y - vB_z),\quad \bar{E}_z = \gamma(E_z + vB_y)$$

$$\bar{B}_x = B_x,\quad \bar{B}_y = \gamma(B_y + \frac{v}{c^2}E_z),\quad \bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y)$$

$$\begin{aligned}
\bar{\mathbf{E}} \cdot \bar{\mathbf{B}} &= E_xB_x + \gamma(E_y - vB_z)\gamma(B_y + \frac{v}{c^2}E_z) + \gamma(E_z + vB_y)\gamma(B_z - \frac{v}{c^2}E_y) \\
&= E_xB_x + \gamma^2(E_yB_y - \frac{v^2}{c^2}B_zE_z + \frac{v}{c^2}E_yE_z - vB_yB_z + E_zB_z - \frac{v^2}{c^2}E_yB_y + vB_yB_z - \frac{v}{c^2}E_yE_z) \\
&= E_xB_x + \gamma^2(1 - \frac{v}{c^2})(E_yB_y + E_zB_z) \\
&= \mathbf{E} \cdot \mathbf{B}
\end{aligned}$$

So it is relativistically invariant

(b) $\mathbf{B} = 0$ but $\mathbf{E} \neq 0$

$$\bar{E}_x = E_x,\quad \bar{E}_y = \gamma(E_y),\quad \bar{E}_z = \gamma(E_z)$$

$$\bar{B}_x = 0,\quad \bar{B}_y = \gamma(\frac{v}{c^2}E_z),\quad \bar{B}_z = \gamma(\frac{v}{c^2}E_y)$$

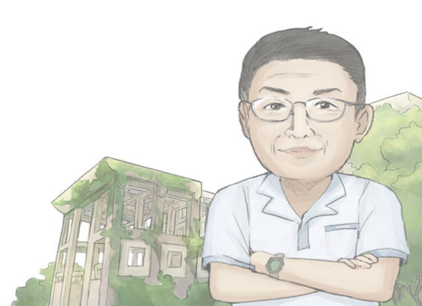
It is impossible to find another system in which the electric field is zero at P

3. Griffith page 424-425, lecture note Chap. 10 pp.14-16

(a)

$$\text{Retarded potentials: } \begin{cases} \text{scalar} \quad V(\mathbf{r},t) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}d\tau' \\ \text{vector} \quad \mathbf{A}(\mathbf{r},t) = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}d\tau' \end{cases} \quad t_r \equiv t - \frac{\mathscr{r}}{c} \text{ (called the retarded time)}$$

(b) $\nabla V = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \nabla\left(\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}\right)d\tau' = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \frac{\mathscr{r}(\nabla\rho) - \rho(\nabla\mathscr{r})}{\mathscr{r}^2}d\tau'$



Using quotient rule: $\nabla\left(\frac{f}{g}\right)=\frac{g\nabla f-f\nabla g}{g^2}$

$$\nabla\rho=\nabla\rho(\mathbf{r}',t_r)=\frac{\partial\rho}{\partial t_r}\nabla t_r=\dot\rho\frac{-1}{c}\nabla\mathscr{r}\,,\quad \nabla\mathscr{r}=\hat{\mathscr{r}}$$

$$\nabla V=\frac{-1}{4\pi\varepsilon_0}\int[\frac{\dot\rho\hat{\mathscr{r}}}{c\mathscr{r}}+\frac{\rho\hat{\mathscr{r}}}{\mathscr{r}^2}]d\tau'$$

$$\nabla\cdot\nabla V=\nabla^2V=\frac{-1}{4\pi\varepsilon_0}\int\nabla\cdot[\frac{\dot\rho\hat{\mathscr{r}}}{c\mathscr{r}}+\frac{\rho\hat{\mathscr{r}}}{\mathscr{r}^2}]d\tau'$$

$$\nabla\cdot[\frac{\dot\rho\hat{\mathscr{r}}}{c\mathscr{r}}+\frac{\rho\hat{\mathscr{r}}}{\mathscr{r}^2}]=\frac{1}{c}\nabla\cdot(\dot\rho\frac{\hat{\mathscr{r}}}{\mathscr{r}})+\nabla\cdot(\rho\frac{\hat{\mathscr{r}}}{\mathscr{r}^2})$$
$$=\frac{1}{c}[\frac{\hat{\mathscr{r}}}{\mathscr{r}}\cdot\nabla\dot\rho+\dot\rho\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}}]+[\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\nabla\rho+\rho\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}]$$

$$\nabla\dot\rho=\nabla\dot\rho(\mathbf{r}',t_r)=\frac{\partial\dot\rho}{\partial t_r}\nabla t_r=\ddot\rho\frac{-1}{c}\nabla\mathscr{r}=-\frac{\ddot\rho}{c}\hat{\mathscr{r}}\quad\text{and}\quad\nabla\rho=\frac{-\dot\rho}{c}\hat{\mathscr{r}}$$

$$\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}}=\frac{1}{\mathscr{r}^2}\quad\text{and}\quad\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}=4\pi\delta^3(\mathscr{r})$$

$$\nabla\cdot[\frac{\dot\rho\hat{\mathscr{r}}}{c\mathscr{r}}+\frac{\rho\hat{\mathscr{r}}}{\mathscr{r}^2}]=\frac{1}{c}[-\frac{\ddot\rho}{c\mathscr{r}}+\frac{\dot\rho}{\mathscr{r}^2}]+[-\frac{1}{\mathscr{r}^2}\frac{\dot\rho}{c}+4\pi\rho\delta^3(\mathscr{r})]$$
$$=-\frac{1}{c^2}\ddot\rho+4\pi\rho\delta^3(\mathscr{r})$$

$$\nabla^2V=\frac{-1}{4\pi\varepsilon_0}\int[-\frac{1}{c^2}\ddot\rho+4\pi\rho\delta^3(\mathscr{r})]d\tau'=\frac{1}{c^2}\int\frac{\ddot\rho}{4\pi\varepsilon_0}d\tau'-\frac{\rho(\mathbf{r},t)}{\varepsilon_0}$$

$$\int\frac{\ddot\rho}{4\pi\varepsilon_0}d\tau'=\int\frac{1}{4\pi\varepsilon_0}\frac{\partial^2\rho}{\partial t_r^2}d\tau'=\frac{\partial^2}{\partial t_r^2}\int\frac{\rho}{4\pi\varepsilon_0}d\tau'=\frac{\partial^2V}{\partial t_r^2}=\frac{\partial^2V}{\partial t^2}$$

$$\nabla^2V=\frac{1}{c^2}\frac{\partial^2V}{\partial t^2}-\frac{\rho(\mathbf{r},t)}{\varepsilon_0}\quad\Rightarrow\quad\nabla^2V-\frac{1}{c^2}\frac{\partial^2V}{\partial t^2}=-\frac{\rho(\mathbf{r},t)}{\varepsilon_0}$$

4. HW 9.38

(a)

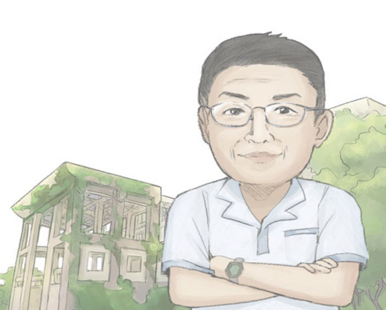
$$\mathbf{E}(s,\phi,z,t)=\frac{A\cos(kz-\omega t)}{s}\hat{\mathbf{s}}\quad\Rightarrow\quad\mathbf{B}(s,\phi,z,t)=\frac{A\cos(kz-\omega t)}{cs}\hat{\boldsymbol{\varphi}}$$

$$\mathbf{S}\equiv\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})=\frac{A^2\cos^2(kz-\omega t)}{\mu_0cs^2}\hat{\mathbf{s}}\times\hat{\boldsymbol{\varphi}}=\frac{A^2\cos^2(kz-\omega t)}{\mu_0cs^2}\hat{\mathbf{z}}$$

(b) new boundary conditions
$$\mathbf{E}(s,\phi,z=0\text{ and }d,t)=\frac{A\sin(kz)}{s}\sin(\omega t)\hat{\mathbf{s}},\ k=\frac{\pi}{d}$$
$$\mathbf{B}(s,\phi,z,t)=\frac{A\cos(kz)}{cs}\cos(\omega t)\hat{\boldsymbol{\varphi}}$$

$$k=\frac{\pi}{d},\lambda=2d\quad\Rightarrow f=\frac{c}{2d}$$

$$\mathbf{S}\equiv\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})=\frac{A^2\cos(kz)\sin(kz)}{\mu_0cs^2}\cos(\omega t)\sin(\omega t)\hat{\mathbf{z}}$$



5. (a)

Incident wave: $\tilde{\mathbf{E}}_I(z,t) = \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \hat{\mathbf{y}}$, $\tilde{\mathbf{B}}_I(z,t) = -\frac{1}{v_1} \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \hat{\mathbf{x}}$

Reflected wave: $\tilde{\mathbf{E}}_R(z,t) = \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{\mathbf{y}}$, $\tilde{\mathbf{B}}_R(z,t) = \frac{1}{v_1} \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{\mathbf{x}}$

Transmitted wave: $\tilde{\mathbf{E}}(z,t) = \tilde{E}_{0T} e^{i(k_2 z - \omega t)} \hat{\mathbf{y}}$, $\tilde{\mathbf{B}}(z,t) = -\frac{1}{v_2} \tilde{E}_{0T} e^{i(k_2 z - \omega t)} \hat{\mathbf{x}}$

(b) Boundary conditions: $\mathbf{E}_1^{//} - \mathbf{E}_2^{//} = 0$ and $\frac{1}{\mu_1}\mathbf{B}_1^{//} - \frac{1}{\mu_2}\mathbf{B}_2^{//} = 0$

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} \quad \text{and} \quad \frac{1}{\mu_1 v_1}(\tilde{E}_{0I} - \tilde{E}_{0R}) - \frac{\tilde{k}_2}{\mu_2 \omega}\tilde{E}_{0T} = 0$$

$$\begin{aligned} \tilde{E}_{0R} &= \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{0I} \\ \tilde{E}_{0T} &= \left(\frac{2}{1+\beta}\right)\tilde{E}_{0I} \end{aligned} \quad \text{where} \quad \beta \equiv \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2$$

$$I_I \equiv \langle \mathbf{S}\cdot\hat{\mathbf{z}} \rangle = \frac{1}{2} v_1 \varepsilon_1 E_{0I}^2 \cos\theta_I, \quad I_T = \frac{1}{2} v_2 \varepsilon_2 E_{0T}^2 \cos\theta_T = \beta\left(\frac{2}{1+\beta}\right)^2 I_I$$

$$T \equiv \frac{I_T}{I_I} = \beta\left(\frac{2}{1+\beta}\right)^2$$